

# Neutrino flavor oscillations in background matter

**Maxim Dvornikov**

Department of Physics, P.O. Box 35, FIN-40014, University of Jyväskylä, Finland;  
IZMIRAN, 142190, Troitsk, Moscow region, Russia

E-mail: dvmaxim@cc.jyu.fi; maxdvo@izmiran.ru

**Abstract.** We study the evolution of two mixed Dirac neutrinos in an external axial-vector field. The dynamics of this system is described in the framework of relativistic wave equations approach on the basis of the known solutions of the Dirac equation in an external field with a given initial condition. The general solution of the initial condition problem, exactly accounting for the external field, is obtained. Then we consider special form of the external field which corresponds to the standard model neutrino interactions with the background matter. In this limit we derive the transition probability and compare it with the previously known results.

The problem of neutrino oscillations in various external fields attracts considerable attention mainly because the amplification of neutrino oscillations in background matter (MSW effect, see Refs. [1, 2]) is the most plausible explanation of the solar neutrino deficit. We discussed neutrino flavor and spin-flavor oscillations in our recent works [3, 4, 5, 6] in frames of the relativistic wave equations approach (classical field theory) which exactly accounts for external fields. The present paper continues our previous efforts.

Let us discuss the evolution of two mixed flavor neutrinos  $\nu_\lambda$ ,  $\lambda = \alpha, \beta$ , interacting with the external field axial-vector field  $f_{\lambda\lambda'}^\mu$ . We assume that the matrix  $(f_{\lambda\lambda'}^\mu)$  is generally not diagonal in the indexes  $\lambda, \lambda'$ . Note that the similar external fields were recently discussed in Ref. [7] (see also references therein). The Lagrangian describing the dynamics of the considered system is

$$\mathcal{L}(\nu_\alpha, \nu_\beta) = \sum_{\lambda=\alpha,\beta} \bar{\nu}_\lambda i\gamma^\mu \partial_\mu \nu_\lambda - \sum_{\lambda,\lambda'=\alpha,\beta} [m_{\lambda\lambda'} \bar{\nu}_\lambda \nu_{\lambda'} + f_{\lambda\lambda'}^\mu \bar{\nu}_\lambda \gamma_\mu^L \nu_{\lambda'}], \quad (1)$$

where  $(m_{\lambda\lambda'})$  is the mass matrix of flavor neutrinos and  $\gamma_\mu^L = \gamma_\mu(1 + \gamma^5)/2$ . Note that the matrices  $(m_{\lambda\lambda'})$  and  $(f_{\lambda\lambda'}^\mu)$  are generally independent. We supply the Lagrangian (1) with the initial conditions,  $\nu_\alpha(\mathbf{r}, 0) = 0$  and  $\nu_\beta(\mathbf{r}, 0) = \xi(\mathbf{r})$ , where  $\xi(\mathbf{r})$  is a given function. This situation is implemented, e.g., if  $\nu_\alpha \equiv \nu_\mu$ ,  $\nu_\beta \equiv \nu_e$  and we study oscillations of solar neutrinos, when only electron neutrinos are presented initially.

To examine the evolution of the system (1) we introduce the mass eigenstates  $\psi_a(\mathbf{r}, t)$ ,  $a = 1, 2$ , by means of the matrix transformation,  $\nu_\lambda = \sum U_{\lambda a} \psi_a$ . In case of two flavor neutrinos the matrix  $(U_{\lambda a})$  is the rotation matrix in the flavor basis,  $U = \exp(-i\sigma_2\theta)$ , where  $\theta$  is the vacuum mixing angle. The effective Lagrangian for the mass eigenstates is obtained from Eq. (1),

$$\mathcal{L}(\psi_1, \psi_2) = \sum_{a=1,2} \bar{\psi}_a (i\gamma^\mu \partial_\mu - m_a) \psi_a - \sum_{a,b=1,2} g_{ab}^\mu \bar{\psi}_a \gamma_\mu^L \psi_b, \quad (2)$$

where  $m_a$  the mass is fermion  $\psi_a$  and  $(g_{ab}^\mu) = U^{-1}(f_{\lambda\lambda'}^\mu)U$  is the external axial-vector field expressed in the mass eigenstates basis.

The Dirac equations which result from Eq. (2) have the form,

$$i\dot{\psi}_a = \mathcal{H}_a \psi_a + V \psi_b, \quad a, b = 1, 2, \quad a \neq b, \quad (3)$$

where  $\mathcal{H}_a = -i\boldsymbol{\alpha}\nabla + \beta m_a + \beta\gamma_\mu^L g_a^\mu$ ,  $V = \beta\gamma_\mu^L g^\mu$ ,  $g_a^\mu = g_{aa}^\mu$  and  $g^\mu = g_{12}^\mu$ . The general solution to Eq. (3) can be presented in the following way (see Refs. [4, 5, 6]):

$$\psi_a(\mathbf{r}, t) = \int \frac{d^3\mathbf{p}}{(2\pi)^{3/2}} e^{i\mathbf{p}\mathbf{r}} \sum_{\zeta=\pm 1} \left[ a_a^{(\zeta)}(t) u_a^{(\zeta)} \exp(-iE_a^{(\zeta)}t) + b_a^{(\zeta)}(t) v_a^{(\zeta)} \exp(+iE_a^{(\zeta)}t) \right], \quad (4)$$

where  $a_a^{(\zeta)}$  and  $b_a^{(\zeta)}$  are the undetermined *non-operator* coefficients, which are generally time dependent. The energy spectrum  $E_a^{(\zeta)}$  in case of non-moving and unpolarized mater, which corresponds to  $\mathbf{g}_a = \mathbf{g} = 0$ , was found in Refs. [8, 9],

$$E_a^{(\zeta)} = \sqrt{\mathbf{p}^2 \left( 1 - \zeta \frac{g_a}{2|\mathbf{p}|} \right)^2 + m_a^2 + \frac{g_a}{2}}. \quad (5)$$

In Eq. (5) we introduce the quantity  $g_a \equiv g_a^0$ . The basis spinors  $u_a^{(\zeta)}$  and  $v_a^{(\zeta)}$  in Eq. (4) are the eigenvectors of the helicity operator  $(\boldsymbol{\Sigma}\mathbf{p})/|\mathbf{p}|$ , with the eigenvalues  $\zeta = \pm 1$ , and can be also found in the explicit form in Refs. [8, 9].

Let us assume the initial wave function in the form,  $\xi(\mathbf{r}) = e^{i\mathbf{k}\mathbf{r}}\xi_0$ , where  $\mathbf{k} = (0, 0, k)$ . If we study relativistic neutrinos with  $k \gg m_{1,2}$ , the normalized basis spinors in Eq. (4) are

$$u^+ = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \quad u^- = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ -1 \\ 0 \\ 1 \end{pmatrix}, \quad v^+ = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 0 \\ -1 \\ 0 \end{pmatrix}, \quad v^- = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ 1 \\ 0 \\ 1 \end{pmatrix}. \quad (6)$$

One can take that  $\xi_0 = u^-$ . It is easy to check that  $(1/2)(1 - \Sigma_3)\xi_0 = \xi_0$ . Therefore  $\xi(\mathbf{r})$  describes a neutrino propagating along the  $z$ -axis, with the spin directed opposite to the particle momentum. Note that the subscript  $a$  is omitted in Eq. (6) since we neglect small terms  $\sim m_a/k \ll 1$ .

With help of the obvious identities,  $\langle u^- | V | u^- \rangle = \langle v^+ | V | v^+ \rangle = g^0$  (all other matrix elements of the potential  $V$  vanish), which result from Eq. (6), as well as using the technique developed in Refs. [5, 6] we get the ordinary differential equations for the functions  $a_a^-(t)$ ,

$$i\dot{a}_a^- = a_b^- g \exp[i(E_a^- - E_b^-)t], \quad a, b = 1, 2, \quad a \neq b, \quad (7)$$

where  $g \equiv g^0$ . Note that the equations for the functions  $b_a^+(t)$  are obtained analogously. The solution to Eq. (7) can be expressed in the form (see, e.g., Refs. [5, 6])

$$\begin{aligned} a_1^-(t) &= F a_1^-(0) + G a_2^-(0), & F &= \left[ \cos \Omega t - i \frac{\omega}{2\Omega} \sin \Omega t \right] \exp(i\omega t/2), \\ a_2^-(t) &= F^* a_2^-(0) - G^* a_1^-(0), & G &= -i \frac{g}{\Omega} \sin \Omega t \exp(i\omega t/2), \end{aligned} \quad (8)$$

where  $\Omega = \sqrt{g^2 + (\omega/2)^2}$  and  $\omega = E_1^- - E_2^-$ .

Using the identity  $(v^+ \otimes v^{+\dagger}) \xi_0 = 0$  [see Eq. (6)] as well as Eqs. (4) and (8) we arrive to the wave function of the neutrino  $\nu_\alpha$ ,

$$\nu_\alpha(z, t) = -i \exp(-i\bar{\mathcal{E}}t + ikz) \sin \Omega t \frac{[g \cos 2\theta + (\omega/2) \sin 2\theta]}{\Omega} \xi_0 + \mathcal{O}\left(\frac{m_a}{k}\right), \quad (9)$$

where  $\bar{\mathcal{E}} = (E_1^- + E_2^-)/2$ . Note that Eq. (9) is the most general one which is valid for the external axial-vector fields  $f_{\lambda\lambda'}^\mu$  of arbitrary strength. Let us however discuss one of the applications of the obtained result. We consider diagonal matrix  $f_{\lambda\lambda'}^\mu = f_\lambda^\mu \delta_{\lambda\lambda'}$  which corresponds to the standard model neutrino interactions with the non-moving and unpolarized background matter, i.e. we take that  $\mathbf{f}_\lambda = 0$ . Note that the values of  $f_\lambda^0$  for various channels of neutrino oscillations can be found in Ref. [10] In this case we have  $g = -\sin 2\theta \Delta f^0$ , where  $\Delta f^0 = (f_\alpha^0 - f_\beta^0)/2$ . In the low density matter limit,  $g_a \ll k$ , one reads  $\omega/2 \approx \Phi(k) + \cos 2\theta \Delta f^0$  [see Eq. (5)], where  $\Phi(k) = \delta m^2/(4k)$  and  $\delta m^2 = m_1^2 - m_2^2$ .

The transition probability can be calculated on the basis of Eq. (9) as

$$P_{\nu_\beta \rightarrow \nu_\alpha}(t) = |\nu_\alpha(z, t)|^2 \approx A \sin^2 \left( \frac{\pi}{L} t \right), \quad (10)$$

where

$$A = \frac{\Phi^2(k) \sin^2(2\theta)}{[\Phi(k) \cos 2\theta + \Delta f^0]^2 + \Phi^2(k) \sin^2(2\theta)}, \quad \frac{\pi}{L} = \sqrt{[\Phi(k) \cos 2\theta + \Delta f^0]^2 + \Phi^2(k) \sin^2(2\theta)}. \quad (11)$$

We can observe that Eqs. (10) and (11) reproduce the famous formula for the neutrino oscillations probability in the background matter (see Refs. [1, 2]).

In conclusion we mention that neutrino flavor oscillations in background matter with non-standard interactions have been studied in frames of the relativistic wave equations approach (classical field theory). Neutrino interactions with matter are known to be equivalent to the presence of an external axial-vector field. In the limit of ultrarelativistic neutrinos interacting with non-moving and unpolarized matter we have received the most general expression for the neutrino wave function [Eq. (9)], which exactly takes into account the external axial-vector field. Using this result and considering standard model neutrino matter interactions we could reproduce the transition probability [Eqs. (10) and (11)] accounting for the resonance enhancement of neutrino oscillations – the MSW effect. Note that with help of the method used in the present work we have improved the results of our recent paper [4]. In contrast to that work, where the case of only low density matter was discussed, now we could obtain the transition probability valid for arbitrary matter density.

## Acknowledgments

The work has been supported by the Academy of Finland under the contract No. 108875. The author is thankful to the Russian Science Support Foundation for a grant and the organizers of the EPS HEP 2007 conference for the invitation.

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